Subspace-Preconditioning for GMRES

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This work is concerned with large-scale discrete ill-posed problems with a square coefficient matrix $A \in \mathbb{R}^{n\times n}$, and demonstrates a new preconditioner for GMRES. We want to solve $Ax = b$, knowing the coefficient matrix A and the measured data b. Due to ill-conditioning of A, as well as measurement errors in b, we need a stabilized solution which is often achieved by Tikhonov regularization

$$x_\lambda = \text{argmin}_x \left\{ \|Ax - b\|_2^2 + \lambda^2 \|Lx\|_2^2 \right\} = \left(A^T A + \lambda^2 L^2 L \right)^{-1} A^T b. \tag{1}$$

We know from the literature that solutions to (1) can be computed by eg., CG-based methods using the least-squares formulation above. Alternatively, applying a Krylov method directly to $\min \|Ax - b\|_2$ or $Ax = b$ has more recently been shown to have an intrinsic regularizing effect. Hanke and Hansen [1] demonstrated how the CGLS algorithm can be modified to take into account the regularizing term $\|Lx\|_2$ by using the A-weighted pseudoinverse $L_A^\dagger$ and solving instead $\min \|AL_A^\dagger y - b\|_2$, using the back transformation $x = L_A^\dagger y + x_0$ where x_0 is the component in a possible nullspace of L.

This approach inhibits the use of GMRES because the modified coefficient matrix $AL_A^\dagger$ is, in general, not square. We present a new approach that takes into account the regularizing term $\|Lx\|_2$ using an oblique projector $P \in \mathbb{R}^{n\times n}$ such that the new transformed coefficient matrix $L_A^{\dagger T} P A L_A^\dagger$, and the right-hand side $L_A^{\dagger T} P b$, can be used with GMRES. The approach can be viewed as running GMRES on a left preconditioned system with the preconditioner $L_A^\dagger L_A^{\dagger T} P$. The proposed preconditioning is especially interesting if eg., A is not given exactly, but only through its effect when applied to a vector, and if A^T is not easily defined. In this case, least-squares algorithms such as CGLS are not directly applicable.

References

[1] M. Hanke and P. C. Hansen, *Regularization methods for large-scale problems,* Surveys Math. Industry, 3 (1993), pp. 253-315