

**Cartan for Beginners:
Differential Geometry via Moving Frames
and Exterior Differential Systems**

Thomas A. Ivey and J.M. Landsberg

Contents

Preface	ix
Chapter 1. Moving Frames and Exterior Differential Systems	1
§1.1. Geometry of surfaces in \mathbb{E}^3 in coordinates	2
§1.2. Differential equations in coordinates	5
§1.3. Introduction to differential equations without coordinates	8
§1.4. Introduction to geometry without coordinates: curves in \mathbb{E}^2	12
§1.5. Submanifolds of homogeneous spaces	15
§1.6. The Maurer-Cartan form	16
§1.7. Plane curves in other geometries	20
§1.8. Curves in \mathbb{E}^3	23
§1.9. Exterior differential systems and jet spaces	26
Chapter 2. Euclidean Geometry and Riemannian Geometry	35
§2.1. Gauss and mean curvature via frames	36
§2.2. Calculation of H and K for some examples	39
§2.3. Darboux frames and applications	42
§2.4. What do H and K tell us?	43
§2.5. Invariants for n -dimensional submanifolds of \mathbb{E}^{n+s}	45
§2.6. Intrinsic and extrinsic geometry	47
§2.7. Space forms: the sphere and hyperbolic space	57
§2.8. Curves on surfaces	58
§2.9. The Gauss-Bonnet and Poincaré-Hopf theorems	61

§2.10. Non-orthonormal frames	66
Chapter 3. Projective Geometry	71
§3.1. Grassmannians	72
§3.2. Frames and the projective second fundamental form	76
§3.3. Algebraic varieties	81
§3.4. Varieties with degenerate Gauss mappings	89
§3.5. Higher-order differential invariants	94
§3.6. Fundamental forms of some homogeneous varieties	98
§3.7. Higher-order Fubini forms	107
§3.8. Ruled and uniruled varieties	113
§3.9. Varieties with vanishing Fubini cubic	115
§3.10. Dual varieties	118
§3.11. Associated varieties	123
§3.12. More on varieties with degenerate Gauss maps	125
§3.13. Secant and tangential varieties	128
§3.14. Rank restriction theorems	132
§3.15. Local study of smooth varieties with degenerate tangential varieties	134
§3.16. Generalized Monge systems	137
§3.17. Complete intersections	139
Chapter 4. Cartan-Kähler I: Linear Algebra and Constant-Coefficient Homogeneous Systems	143
§4.1. Tableaux	144
§4.2. First example	148
§4.3. Second example	150
§4.4. Third example	153
§4.5. The general case	154
§4.6. The characteristic variety of a tableau	157
Chapter 5. Cartan-Kähler II: The Cartan Algorithm for Linear Pfaffian Systems	163
§5.1. Linear Pfaffian systems	163
§5.2. First example	165
§5.3. Second example: constant coefficient homogeneous systems	166
§5.4. The local isometric embedding problem	169

§5.5. The Cartan algorithm formalized: tableau, torsion and prolongation	173
§5.6. Summary of Cartan's algorithm for linear Pfaffian systems	177
§5.7. Additional remarks on the theory	179
§5.8. Examples	182
§5.9. Functions whose Hessians commute, with remarks on singular solutions	189
§5.10. The Cartan-Janet Isometric Embedding Theorem	191
§5.11. Isometric embeddings of space forms (mostly flat ones)	194
§5.12. Calibrated submanifolds	197
Chapter 6. Applications to PDE	203
§6.1. Symmetries and Cauchy characteristics	204
§6.2. Second-order PDE and Monge characteristics	212
§6.3. Derived systems and the method of Darboux	215
§6.4. Monge-Ampère systems and Weingarten surfaces	222
§6.5. Integrable extensions and Bäcklund transformations	231
Chapter 7. Cartan-Kähler III: The General Case	243
§7.1. Integral elements and polar spaces	244
§7.2. Example: Triply orthogonal systems	251
§7.3. Statement and proof of Cartan-Kähler	254
§7.4. Cartan's Test	256
§7.5. More examples of Cartan's Test	259
Chapter 8. Geometric Structures and Connections	267
§8.1. G -structures	267
§8.2. How to differentiate sections of vector bundles	275
§8.3. Connections on \mathcal{F}_G and differential invariants of G -structures	278
§8.4. Induced vector bundles and connections on induced bundles	283
§8.5. Holonomy	286
§8.6. Extended example: Path geometry	295
§8.7. Frobenius and generalized conformal structures	308
Appendix A. Linear Algebra and Representation Theory	311
§A.1. Dual spaces and tensor products	311
§A.2. Matrix Lie groups	316
§A.3. Complex vector spaces and complex structures	318

§A.4. Lie algebras	320
§A.5. Division algebras and the simple group G_2	323
§A.6. A smidgen of representation theory	326
§A.7. Clifford algebras and spin groups	330
Appendix B. Differential Forms	335
§B.1. Differential forms and vector fields	335
§B.2. Three definitions of the exterior derivative	337
§B.3. Basic and semi-basic forms	339
§B.4. Differential ideals	340
Appendix C. Complex Structures and Complex Manifolds	343
§C.1. Complex manifolds	343
§C.2. The Cauchy-Riemann equations	347
Appendix D. Initial Value Problems	349
Hints and Answers to Selected Exercises	355
Bibliography	363
Index	371