

# Index

- $II$ , Euclidean second fundamental form, 46  
 $II$ , projective second fundamental form, 77  
 $|II_{M,x}|$ , 80  
 $III$ , projective third fundamental form, 96  
 $III^v$ , 129  
 $\Gamma(E)$ , smooth sections of  $E$ , 335  
 $\Gamma_{\alpha,i}^\beta$ , 277  
 $\Delta r_{\alpha\beta\gamma}^\mu$ , 95  
 $\Lambda^2 V$ , 313  
 $\Lambda^k V$ , 314  
 $\Xi_A$ , characteristic variety of a tableau, 157  
 $\Omega^k(M), \Omega^*(M)$ , 336  
 $\Omega^k(M, V)$ , 338  
 $\Omega^{(p,q)}(M)$ , 345  
 $\delta_\sigma(X)$ , secant defect, 129  
 $\delta_\tau(X)$ , tangential defect, 129  
 $\delta_*$ , dual defect, 120  
 $\phi^*$ , pullback by  $\phi$ , 337  
 $\phi_*$ , pushforward by  $\phi$ , 337  
 $\kappa_g$ , 59  
 $\kappa_n$ , 60  
 $\tau(X)$ , tangential variety, 86  
 $\tau(Y, X)$ , 131  
 $\tau_g$ , 60  
 $A^{(1)}$ , 147  
 $A^{(l)}$ , 147  
 $\text{Ann}(v)$ , 129  
 $ASO(2)$ , 12  
 $ASO(3)$ , 23  
     as space of frames, 24  
 Baseloc  $|II_{M,x}|$ , 80  
 $C^\infty(M)$ , 335  
 $c_k$ , codimension of polar space, 256  
 $Cl(V, Q)$ , 331  
 $d$ , exterior derivative, 337  
 $\underline{d}^k$ , 97  
 $\det$ , 102  
 $\det$ , 315  
 $\mathbb{E}^3$ , Euclidean three-space, 2  
 $E_6$ , exceptional Lie group, 102  
 $\text{End}(V)$ , 312  
 $\mathcal{F}(M)$ , 49  
 $\mathcal{F}^1$   
     Euclidean, 37  
     projective, 78  
 $F_4$ , exceptional Lie group, 102  
 $F_4$ , the differential invariant, 107  
 $F_k$ , 108  
 $\mathbb{FF}^k$ , 97  
 $|\mathbb{FF}^k|$ , 97  
 $\mathfrak{g}$ , Lie algebra of Lie group  $G$ , 17  
 $G_2$ , exceptional Lie group, 323  
 $G(k, V)$ , Grassmannian, 72  
 $\mathbf{G}(n, m)$ , 198  
 $\mathbf{G}(n, T\Sigma)$ , 177  
 $GL(V)$ , 316  
 $Gr(k, V)$ , orthogonal Grassmannian, 75  
 $H^{0,2}(A)$ , 175  
 $H^{i,j}(A)$ , 180  
 $\mathcal{H}^{i,j}(\mathfrak{g})$ , 283  
 $\text{Hol}_u^\theta$ , 287  
 $\text{Hom}(V, W)$ , 312  
 $\mathcal{I}$ , differential ideal, 340  
 $\mathcal{I}^k$ ,  $k$ -th homogeneous component of  $\mathcal{I}$ , 340  
 $I^{(1)}$ , derived system, 216  
 $(I, J)$ , linear Pfaffian system, 164  
 $J(Y, Z)$ , join of varieties, 86  
 $\mathcal{K}(V)$ , 330  
 $\mathcal{L}_X$ , Lie derivative, 339  
 $m_x$ , functions vanishing at  $x$ , 335  
 $O(V, Q)$ , orthogonal group, 317

- $(p, q)$ -forms, 345
- $[R_\theta]$ , 282
- $S^2V$ , 313
- $S^kV$ , 314
- $s_k$ 
  - characters of a tableau, 154
  - characters of an EDS, 258
- $\mathbb{S}_m$ , spinor variety, 106
- Singloc  $|II_{M,x}|$ , 80
- $SL(V)$ ,  $SL_n$ , special linear group, 317
- $SO(V, Q)$ , special orthogonal group, 317
- $Sp(V, \omega)$ , symplectic group, 317
- $SU(n)$ , special unitary group, 319
- $\mathbb{T}(V)$ , 273
- $TM$ , tangent bundle, 335
- $T^*M$ , cotangent bundle, 335
- $T_xM$ , tangent space, 335
- $T_x^*M$ , cotangent space, 335
- $U(n)$ , unitary group, 319
- $V_{\mathbb{C}}$ , complexification of  $V$ , 343
- $X_{\text{smooth}}$ , 82
- $[X, Y]$ , 336
- $\lrcorner$ , interior product, 315
- $\nabla$ , 277
- $\otimes$ , tensor product, 312
- $\sharp$ , 53
- $\{ \}$ , linear span, 340
- $\{ \}_{\text{alg}}$ , 340
- $\{ \}_{\text{diff}}$ , 340
- abuse of notation, 29, 72, 170
- adjoint representation, 321
- affine connection, 285
- affine tangent space, 76
- algebraic variety, 82
  - degree of, 82
  - dimension of, 82
  - general point of, 83
  - ideal of, 82
- almost complex manifold, 274, 282, 344
- almost complex structure, 344
- almost symplectic manifold, 274
- Ambrose-Singer Theorem, 290
- apparent torsion, 165
- arclength parameter, 14
- associated hypersurface, 124
- associated varieties, 123
- associative submanifolds, 201, 265
- associator, 325
- asymptotic directions, 80
- asymptotic line, 60, 226, 238
- Bäcklund transformations, 235–241
- Bäcklund's Theorem, 237
- basic differential form, 339
- Bertini Theorem, 112
  - higher-order, 112
- Bertrand curve, 26
- Bezout's Theorem, 82
- Bianchi identities, 53–54
- Bonnet surface, 44, 231
- Burger's equation, 208, 232
- calibrated submanifold, 198
- calibration, 197
  - associative, 201
  - Cayley form, 202
  - coassociative, 201
  - special Lagrangian, 200
- canonical system
  - on Grassmann bundle, 177
  - on space of jets, 28
- Cartan geometry, 296
- Cartan integer, 156, 179
- Cartan Lemma, 314
- Cartan system, 209
- Cartan's algorithm for linear Pfaffian systems, 178
- Cartan's five variables paper, 217
- Cartan's Test, 256
- Cartan-Dieudonné Theorem, 331
- Cartan-Janet Theorem, 192
- Cartan-Kähler Theorem, 254–256
  - for linear Pfaffian systems, 176
  - for tableaux, 156
  - Goldschmidt version, 181
- catenoid, 43
- Cauchy problem, 349
- Cauchy-Kowalevski form, 350
- Cauchy-Kowalevski Theorem, 243, 351
- Cauchy-Riemann equations, 347
  - tableau, 144, 156
- Cayley submanifold, 202
- character of a tableau, 156
- characteristic hyperplane, 181
- characteristic systems (Monge), 213
- characteristic variety, 157
  - dimension and degree of, 159
- characteristics
  - Cauchy, 205, 259
  - quotient by, 210
  - confounded, 213
  - first-order, 214
  - method of, 207–208
  - Monge, 213
- characters, 258
  - of linear Pfaffian system, 179
  - of tableau, 154
- Chebyshev net, 227
- Christoffel symbols, 277
- Clifford algebras, 331
  - fundamental lemma of, 332
- Clifford torus, 58
- co-roots, 329

- coassociative submanifold, 201
- Codazzi equation
  - for Darboux frames, 43
  - matrix form, 49
- codimension, 245
- coisotropic hypersurface, 124
- complete intersection, 140
- complex characteristic variety, 158
- complex contact structure, 348
- complex manifold, 343, 344
- complex structure, 318, 344
- complexification
  - of a real vector space, 318
- cone, 44
  - characterization of, 125
  - over a variety, 86
- connection
  - affine, 285
  - on coframe bundle, 278–283
  - on induced vector bundles, 284
  - on vector bundle, 277
  - symmetric, 285
- connection form, 279
- conormal space, of submanifold in  $\mathbb{P}^N$ , 77
- contact manifold, 33
- contact system
  - on space of jets, 28
- contact, order of, 83
- cotangent
  - bundle, 335
  - space, 335
- covariant differential operator, 54, 277
- cubic form, 94
- curvature
  - Gauss, 38
    - geometric interpretation of, 47
    - in coordinates, 4
  - mean, 38
    - geometric interpretation of, 68
    - in coordinates, 4
  - of curve in  $\mathbb{E}^2$ , 14
  - of curve in  $\mathbb{E}^3$ , 25
  - of  $G$ -structure, 280
  - Ricci, 53, 262
  - scalar, 53, 262, 266, 330
  - sectional, 53
  - traceless Ricci, 330
  - Weyl, 330
- curvature-line coordinates, 188
- curve
  - arclength parameter, 14
  - Bertrand, 26
  - regular, 13
  - speed of, 14
- curve in  $\mathbb{E}^2$ 
  - curvature, 14
  - osculating circle, 14
- curve in  $\mathbb{E}^3$ 
  - curvature, 25
  - differential invariants, 25–26
  - torsion, 25
- cylinder, 44
- Darboux
  - integrable, 218, 239
  - method of, 217–222
  - semi-integrable, 222
- Darboux frame, 42
- Darboux's Theorem, 32
- de Rham Splitting Theorem, 289
- decomposable tensor, 312
- derived flag, 216
- derived system, 216
- determinant
  - of linear endomorphism, 315
- developable surface, 40
- differential form, 336
  - basic, semi-basic, 339
  - closed, 338
  - homogeneous, 340
  - left-invariant, 17
  - vector-valued, 338
- differential ideal, 340
- differential invariant
  - Euclidean, 3
- dual basis, 311
- dual variety, 87, 118
  - defect of, 120
  - reflexivity, 119
- dual vector space, 311
- Dupin
  - cyclides of, 361
  - theorem of, 253
- $e$ -structure, 304
- embedded tangent space, 76
- Engel structure, 217
- equivalent
  - $G$ -structures, 275
  - webs, 268
- Euclidean group, 23
- Euler characteristic, 62
- exterior derivative, 337–338
- exterior differential system, 29
  - hyperbolic, 214–215
  - linear Pfaffian, 164
  - Pfaffian, 341
  - symmetries, 204–205
  - with independence condition, 27
- face of calibration, 199
- first fundamental form (Riemannian), 46
- first-order adapted frames (Euclidean), 45

- flag
  - $A$ -generic, 154
  - complete, 85
  - derived, 216
  - partial, 85
- flag variety, 85, 316
- flat
  - $G$ -structure, 275
  - 3-web, 268
  - path geometry, 296
  - Riemannian manifold, 52
    - isometric immersions of, 194
  - surface, 41
- flow of a vector field, 6
- flowbox coordinates, 6
- flowchart for Cartan's algorithm, 178
- focal hypersurface, 89
- focal surface, 237, 266
- frame
  - Darboux, 42
- frame bundle
  - general, 49
  - orthonormal, 50
- Frenet equations, 25
- Frobenius ideal, 11
- Frobenius structure, 308
- Frobenius system
  - tableau of, 146
- Frobenius Theorem, 10–12, 30
  - proof, 30
- Fubini cubic form, 94
- Fubini forms, 94, 107
- Fulton-Hansen Theorem, 130
- fundamental form
  - effective calculation of, 97
  - $k$ -th, 97
  - prolongation property of, 97
  - via spectral sequences, 98
- $G$ -structure, 267–275
  - 1-flat, 280
  - 2-flat, 281
  - curvature, 280, 282
  - definition, 274
  - flat, 275
  - prolongation, 281
- $G/H$ -structure of order two, 296
- Gauss curvature
  - geometric interpretation of, 47
  - in coordinates, 4
  - via frames, 36–38
- Gauss equation, 47
- Gauss image, 77
  - characterization of, 93
- Gauss map
  - algebraic, 55
  - Euclidean, 46
  - projective, 77
    - varieties with degenerate, 89
- Gauss' theorema egregium, 48
- Gauss-Bonnet formula, 64
- Gauss-Bonnet theorem, 62
  - for compact hypersurfaces, 64
  - local, 60
- Gauss-Bonnet-Chern Theorem, 65
- general point, 83
- generalized conformal structure, 309
- generalized Monge system, 139
- generic point, 83
- geodesic, 59
  - of affine connection, 285
- geodesic curvature, 59
- geodesic torsion, 60
- Grassmann bundle, 177
  - canonical system on, 177
- Grassmannian, 72, 316
  - isotropic, 84
  - tangent space of, 73
- half-spin representation, 107
- Hartshorne's conjecture, 140
- heat equation, 350
- helicoid, 39
- Hermitian form, 319
- Hermitian inner product, 319
- hexagonality, 271
- higher associated hypersurface, 124
- holomorphic map, 345
- holonomy, 286–295
- holonomy bundle, 287
- holonomy group, 287
- homogeneous space, 15
- Hopf differential, 230
- horizontal curve, 287
- horizontal lift, 287
- hyperbolic space, 58
  - isometric immersions of, 197
- hyperplane section of a variety, 88
- hypersurfaces in  $\mathbb{E}^N$ 
  - fundamental theorem for, 55
- ideal
  - algebraic, 340
  - differential, 340
  - Frobenius, 11
- incidence correspondence, 88
- independence condition, 27
- index of a vector field, 61
- index of relative nullity, 80
- induced vector bundle, 283
- initial data, 349
- initial value problem, 349
- integrable extension, 232
  - via conservation law, 233

- integral
  - intermediate/general, 219
- integral curve, 5
- integral element, 27
  - Kähler-ordinary, 245
  - Kähler-regular, 249
  - ordinary, 256
- integral manifold, 27, 29
- interior product, 315
- involutive
  - integral element, 256
  - linear Pfaffian system, 176
  - tableau, 155
- isometric embedding, 169–173
- isothermal coordinates, 57
  - existence of, 185
- isotropic Grassmannian, 84
- isotropy representation, 16
  
- Jacobi identity, 320
- jets, 27
- join of varieties, 86
  
- Kähler manifold, 199
- KdV equation, 234, 236
  - prolongation algebra, 235
- Killing form, 323
  
- Laplace system
  - tableau for, 157
- Laplace's equation, 223
- Laplacian, 56
- left action, 15
- left-invariant
  - differential form, 17
  - vector field, 17, 320
- level, 155
- Lie algebra, 320
  - of a Lie group, 17
  - semi-simple, 327
  - simple, 327
- Lie bracket, 336
- Lie derivative, 339
- Lie group, 316
  - linear representation of, 316
  - matrix, 16, 316–318
  - Maurer-Cartan form of, 17
- lift, 16
  - first-order adapted, 37
- line congruence, 237
- line of curvature, 60, 253
  - isothermal coordinates along, 188
- linear map, 311
  - transpose/adjoint of, 312
- linear normality
  - Zak's theorem on, 128
- linear Pfaffian systems, 164
  - Cartan's algorithm for, 178
  - involutivity, 176
  - linear projection of variety, 88
  - linear syzygy, 111
  - linear Weingarten surface, 183
  - Liouville's equation, 218, 237
  - locally ruled variety, 89
  - locally symmetric, 290
  
- majorants, 150
- manifold
  - contact, 33
  - restraining, 255
  - symplectic, 31
- matrix Lie groups, 316–318
- Maurer-Cartan equation, 18
- Maurer-Cartan form
  - of a matrix Lie group, 17
  - of an arbitrary Lie group, 17
- maximal torus, 327
- mean curvature
  - geometric interpretation of, 68
  - in coordinates, 4
  - via frames, 36–38
- mean curvature vector, 69
- minimal hypersurfaces, 266
- minimal submanifold, 197
- minimal surface, 68, 228–229
  - Riemannian metric of, 186
- minimizing submanifold, 197
- minuscule variety, 104
- modified KdV equation, 234
- Monge's method, 224
- Monge-Ampère
  - equation, 222
  - system, 223
- moving frame, 4
  - adapted, 12
- multilinear, 312
- multiplicity of intersection, 83
- musical isomorphism, 53
  
- Newlander-Nirenberg Theorem, 345
- Nijenhuis tensor, 346
- non-characteristic initial data, 157
- nondegenerate quadratic form, 322
- normal bundle, 46, 66
- normal curvature, 60
- normal space, of submanifold in  $\mathbb{P}^N$ , 77
  
- octonions, 324–326
- orthogonal Grassmannian, 75
- orthogonal group, 317
- orthogonal involutive Lie algebra, 291
- osculating circle, 14
- osculating hypersurface, 109, 111
- osculating quadric hypersurface, 109

- parabolic subgroup, 84, 104
- parallel surfaces, 225
- parallel transport, 287
- path geometry, 295–308
  - definition of, 295, 298
  - dual, 297
  - flat, 296
- Pfaff's Theorem, 33
- Pfaffian, 322
- Pfaffian system, 341
  - linear, 164
- Picard's Theorem, 5, 10
- Poincaré-Hopf Theorem, 62
- point transformation, 295
- polar spaces, 246–248
- principal curvatures, 39
- principal framing, 42
- principal symbol, 145
- projective differential invariants
  - in coordinates, 108
- projective second fundamental form, 77
  - coordinate description of, 81
  - frame definition of, 79
- projective structure, 286
- prolongation, 147, 177, 214, 220
  - of a  $G$ -structure, 281
- prolongation property, 97
  - strict, 105
- prolongation structures, 233
- pseudospherical surfaces, 226–228
  - Bäcklund transformation for, 237
  - of revolution, 227
- pullback, 337
- pushforward, 337
- rank
  - of a Lie algebra, 327
  - of a Pfaffian system, 341
  - of a tensor, 313
- rational homogeneous variety, 83
- reductive
  - Lie group/Lie algebra, 327
- refined third fundamental form, 129
- regular curve, 13
- regular second-order PDE, 174
- relative tangent star, 131
- representation
  - isotropy, 16
  - of Lie algebra, 320
  - of Lie group, 316
- restraining manifold, 255
- retracting space, 209
- Ricci curvature, 53, 262
- Riemann curvature tensor, 52–55, 273
- Riemann invariant, 217
- Riemann surface, 346
- Riemannian geometry, 271–273
  - fundamental lemma, 50–51, 273
- Riemannian manifold, 47
  - flat, 52
- Riemannian metric, 46, 47
- right action, 15
- root, 328
- root system, 328
- ruled surface, 41
- ruled variety, 113
- $S$ -structure, 309
- scalar curvature, 53, 262, 266, 330
- Schur's Lemma, 317
- Schwarzian derivative, 22
- secant defect, 129
- secant variety, 86
- second fundamental form
  - base locus of, 80
  - Euclidean, 46
  - projective, 77
  - singular locus of, 80
- second-order PDE
  - characteristic variety, 182
  - classical notation, 174
  - tableau, 175
- section
  - of vector bundle, 335
- sectional curvature, 53
- Segre product of varieties, 84
  - fundamental forms of, 101
- Segre variety, 84, 159
  - fundamental forms of, 100
- semi-basic form, 339
- semi-Riemannian manifold, 274
- semi-simple Lie algebra, 327
- Severi variety, 102
  - fundamental form of, 103
  - Zak's theorem on, 128
- signature
  - of quadratic form, 322
- simple Lie algebra, 327
- sine-Gordon equation, 223, 226, 235
- singular solutions, 191
- space form, 57
  - isometric immersions of, 194
- special Lagrangian submanifolds, 200, 265
- special linear group, 317
- special orthogonal group, 317
- special unitary group, 319
- Spencer cohomology, 180
- spin representation, 106, 107
- spinor variety, 85, 106
- stabilizer type, 282
- submanifold
  - associative, 265
  - Lagrangian, 185, 264
  - special Lagrangian, 200, 265

- surface
  - Bonnet, 44
  - catenoid, 43
  - cone, 44
  - constant mean curvature, 229–231
  - cylinder, 44
  - developable, 40
  - flat, 41
  - focal, 237, 266
  - helicoid, 39
  - isothermal coordinates on, 57
  - linear Weingarten, 183
  - minimal, 68, 228
  - of revolution, 41, 227
  - parallel, 225
  - pseudospherical, 226
  - ruled, 41
  - warp of, 4
  - Weingarten, 224, 261
  - with degenerate Gauss image, 91
- symbol mapping, 157
- symbol relations, 145, 174
- symmetric connection, 285
- symmetric Lie algebra, 291
- symmetric space, 290
- symmetries, 241
- symplectic form, 32, 185, 199, 212, 264, 317
- symplectic group, 317
- symplectic manifold, 31
  
- tableau, 145
  - determined, 158
  - of linear Pfaffian system, 174
  - of order  $p$ , 147
- tangent
  - bundle, 335
  - space, 335
- tangent star, 86
- tangential defect, 129
  - critical, 135
- tangential surface, 40
- tangential variety, 86
  - dimension of, 128
- tautological EDS
  - for torsion-free  $G$ -structures, 293
- tautological form
  - for coframe bundle, 49
- tensor product, 312
- Terracini's Lemma, 87
- third fundamental form
  - projective, 96
- torsion
  - of connection, 279
  - of curve in  $\mathbb{E}^3$ , 25
  - of  $G$ -structure, 280
  - of linear Pfaffian system, 165, 175
- transformation
  - Bäcklund, 232, 236
  - Cole-Hopf, 232, 238
  - fractional linear, 20
  - Lie, 231
  - Miura, 234
- triangulation, 61
- triply orthogonal systems, 251–254
  
- umbilic point, 39
- uniruled complex manifold, 310
- uniruled variety, 113
- unitary group, 319
  
- variation of Hodge structure, 189
- variety
  - algebraic, 82
  - dual, 87, 118
  - flag, 85
  - miniscule, 104
  - rational homogeneous, 83
  - ruled, 113
  - secant, 86
  - Segre, 84
  - spinor, 85, 106
  - tangential, 86
  - uniruled, 113
  - Veronese, 85
- vector bundle
  - induced, 283
- vector field, 335
  - flow of a, 6
  - left-invariant, 17
- Veronese embedding, 85
- Veronese re-embedding, 85, 109
- Veronese variety, 85
  - fundamental forms of, 99
- vertical vector, 339
- volume form, 46
  
- Waring problems, 313
- warp of a surface, 4
- wave equation, 203, 349
- web, 267
  - hexagonality of, 271
- wedge product, 314
  - matrix, 18
- Weierstrass representation, 228–229
- weight, 327
  - highest, 329
  - multiplicity of, 327
- weight diagram for invariants, 305
- weight lattice, 329
- weight zero invariant, 300
- Weingarten equation, 224
- Weingarten surface, 224, 261
- Weyl curvature, 330
- Wirtinger inequality, 199

Zak's theorem  
  on linear normality, 128  
  on Severi varieties, 128  
  on tangencies, 131